IN112: Propositional logic semantics

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1. **Inductive definitions**

    1. give an inductive definitions of natural numbers
    2. give an inductive definitions of binary trees
       Prove the following property of binary trees: "the number n of nodes in a binary tree of height h is at least $n = h$ and at most $n = 2^h - 1$ where h is the depth of the tree".

2. **Natural language sentences modelling**

    Use propositional language to model the following declarative sentences. In each case, define the signature of your language.

    1. it is raining and it is cold.
    2. if he eats too much, he will be sick.
    3. it is sunny but it is cold.
    4. if it is cold, I take my jacket.
    5. I take either a jacket, either an umbrella.
    6. it is not raining.
    7. in autumn, if it is cold then I take a jacket.
    8. in winter, I take a jacket only if it is cold.
    9. if Peter does not forget to book tickets, we will go to theater.
    10. if Peter does not forget to book tickets and if we find a baby-sitter, we will go to theater.
    11. he went, although it was very hot, but he forgot his water bottle.
    12. when I am nervous, I practise yoga or relaxation. Someone practising yoga also practises relaxation. So when I do not practise relaxation, I am calm.
    13. my sister wants a black and white cat.

3. **Using algorithms to show validity**

    Show that formula 12 of exercise 2 is valid, not valid, consistent or not

    (a) by using truth tables
    (b) by using equivalent formulas
    (c) by translating it in CNF

4. **A crime to solve**

    Three persons, A, B and C are accused of a crime. They respectively declare to the police officer interrogating them:

    - DA: "B is guilty and C is innocent"
    - DB: "if A is guilty, then C is also guilty"
    - DC: "I am innocent, but at least one of the other persons is guilty"

    Use propositional logic to model the previous sentences and answer the following questions:

    (a) are the three declarations compatible?
    (b) is there a declaration that can be deduced from the other two?
    (c) if A, B and C are innocent, who lies?
    (d) if A, B and C are telling the truth, who is guilty?
    (e) if only innocent people tell the truth, who is guilty?

The following exercises will not be done during lecture. In particular, exercises 6 and 7 are rather theoretical.

5. **A dumb machine**

   We want to build a machine that collects packages and:

   - consolidates them with tape and puts a tag on them, or
   - paints them red and puts a tag on them.

   1. choose a propositional language and write a formula that expresses the expected function of the machine. Let φ_A be this formula.
   2. to build the machine, we only have one machine which consolidates with tape, one machine which puts tags and one machine which paints the packages red.

      Using the language previously defined, write a formula that expresses the function realised with the available machines. Let φ_r be this formula.
   3. prove that the realized function corresponds to the expected function.

6. **Connectors and truth values**

   Let us suppose that $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined using $\neg$ and $\vee$:

   - $\varphi \wedge \psi =_{def} \neg(\neg\varphi \vee \neg\psi)$
   - $\varphi \rightarrow \psi =_{def} \neg\varphi \vee \psi$
   - $\varphi \rightarrow \psi =_{def} \neg(\neg(\neg\varphi \vee \psi) \vee \neg(\varphi \vee \neg\psi))$

   Prove the following properties:

   1. $\llbracket \varphi \wedge \psi \rrbracket_{\mathcal{I}} = T$ iff ($\llbracket \varphi \rrbracket_{\mathcal{I}} = T$ and $\llbracket \psi \rrbracket_{\mathcal{I}} = T$)
   2. $\llbracket \varphi \rightarrow \psi \rrbracket_{\mathcal{I}} = T$ iff (($\llbracket \varphi \rrbracket_{\mathcal{I}} = T$ and $\llbracket \psi \rrbracket_{\mathcal{I}} = T$) or $\llbracket \varphi \rrbracket_{\mathcal{I}} = F$)
   3. $\llbracket \varphi \leftrightarrow \psi \rrbracket_{\mathcal{I}} = T$ iff $\llbracket \varphi \rrbracket_{\mathcal{I}} = \llbracket \psi \rrbracket_{\mathcal{I}}$

7. **Let's prove theorems!**

   Prove:

   1. the replacement theorem and find a counterexample to its reciprocal proposition,
   2. the four lemmas exposed during the lecture,
   3. the deduction theorem,
   4. the substitution theorem.